

# Statistical sum in the CFT driven cosmology

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## Abstract

The path integration technique recently developed for the statistical sum of the microcanonical ensemble in cosmology is applied to the calculation of the one-loop preexponential factor in the cosmological model generated by a conformal field theory with a large number of quantum species – the model of initial conditions possibly related to the resolution of the cosmological constant and landscape problems. The result is obtained for the family of background cosmological instantons with one oscillation of the FRW scale factor. The magnitude of the prefactor is analytically and numerically estimated for fields of various spins conformally coupled to gravity, which justifies the validity of semiclassical expansion for this family of cosmological instantons.

## 1. Introduction

In this paper we apply the path integration technique recently developed for the statistical sum of the generic time-parametrization invariant gravitational system with the FRW metric [1] to a particular case of the cosmology generated by the conformal field theory (CFT) with a large number of quantum species. This problem is motivated by the recently suggested model of initial conditions in cosmology in the form of the microcanonical density matrix [2, 3]. When applied to cosmology with a large number of fields conformally coupled to gravity, this theory can be important within the cosmological constant and dark energy problems. In particular, its statistical ensemble is bounded to a finite range of values of the effective cosmological constant, it generates an inflationary stage and is potentially capable of generating the cosmological acceleration phenomenon within the so-called Big Boost scenario [4].

As shown in [3], for a spatially closed cosmology with  $S^3$ -topology the microcanonical statistical sum can be represented by the Euclidean quantum gravity path integral,

$$Z = \int_{\text{periodic}} D[g_{\mu\nu}, \phi] e^{-S[g_{\mu\nu}, \phi]}, \quad (1.1)$$

over the metric  $g_{\mu\nu}$  and matter fields  $\phi$  which are periodic on the Euclidean spacetime with a compactified time  $\tau$  (of  $S^1 \times S^3$  topology). The FRW metric arises in this path integral as the set of major collective variables of cosmology. Under the decomposition of the full set of  $g_{\mu\nu}(x), \phi(x)$  into the minisuperspace FRW sector

$$ds^2 = N^2(\tau) d\tau^2 + a^2(\tau) d^2\Omega^{(3)}, \quad (1.2)$$

and inhomogeneous “matter” fields  $\Phi(x) = (\phi(x), \psi(x), A_\mu(x), h_{\mu\nu}(x), \dots)$  on the background of this metric the path integral can be cast into the form of an integral over a minisuperspace lapse function  $N(\tau)$  and a scale factor  $a(\tau)$ ,

$$Z = \int D[a, N] e^{-\Gamma[a, N]}, \quad (1.3)$$

$$e^{-\Gamma[a, N]} = \int D\Phi(x) e^{-S[a, N; \Phi(x)]}. \quad (1.4)$$

Here,  $\Gamma[a, N]$  is the Euclidean effective action of the fields  $\Phi$  (which include also the metric perturbations  $h_{\mu\nu}$ ) on the FRW background, and  $S[a, N; \Phi(x)] \equiv S[g_{\mu\nu}, \phi]$  is the original action rewritten in terms of this minisuperspace decomposition. It is important that this representation is not a minisuperspace approximation, when all the fields  $\Phi(x)$  are frozen out. Rather this is the disentangling of the collective degrees of freedom from the configuration space, the rest of which effectively manifest itself in terms of this effective action.

Semiclassically the integral (1.3) is dominated by the contribution of the saddle point – the periodic solution of the effective Friedmann equation of the *Euclidean* gravity theory  $\delta\Gamma/\delta N(\tau) = 0$ ,<sup>1</sup>

$$Z = P \exp(-\Gamma_0). \quad (1.5)$$

Here  $\Gamma_0 = \Gamma[a, N]$  is taken at this solution and  $P$  is the preexponential factor accumulating perturbation corrections of the semiclassical expansion. For the CFT driven cosmology the instanton solutions were obtained in [2] as a function of the primordial cosmological constant, whereas the calculational technique for the one-loop prefactor  $P$  was developed in [1].

Here we calculate this prefactor for the set of instantons in a special range of the cosmological constant in which they have a one-fold nature – a single oscillation of the scale factor  $a(\tau)$  between its minimal and maximal values. In Sect.2 we give the description of the full set of instanton solutions of [2] and compare them with those of [5] – the paper which also considers trace anomaly driven cosmology but differs from our work by setting the problem and the choice of the field-theoretical model. Then after a brief formulation of the calculational method of [1] in Sect.3 we calculate the one-loop prefactor  $P$  in Sect.4. There we begin with the description of the above mentioned set of one-fold instantons and their various characteristics and then estimate the magnitude of  $P$  for quantum fields of various spins. The paper is accomplished with concluding remarks in Conclusions.

## 2. Microcanonical instantons in the CFT driven cosmology

In the theory with a primordial cosmological constant  $\Lambda$  and a large number of free (linear) fields  $\phi$  conformally coupled to gravity – conformal field theory (CFT),

$$S[g_{\mu\nu}, \phi] = -\frac{1}{16\pi G} \int d^4x g^{1/2} (R - 2\Lambda) + S_{CFT}[g_{\mu\nu}, \phi], \quad (2.1)$$

the effective action  $\Gamma[a, N]$  is dominated by the contribution of these fields because they simply outnumber the non-conformal fields including, in particular, the graviton. Then this quantum effective action is exactly calculable as a *functional* of histories  $(a(\tau), N(\tau))$  by the conformal transformation converting (1.2) into the static (Einstein Universe) metric with  $a = \text{const}$  [6, 7, 8]. The result reads [2]

$$\Gamma[a, N] = \oint d\tau N \mathcal{L}(a, a') + F(\eta), \quad (2.2)$$

$$\eta = \oint d\tau \frac{N}{a}, \quad (2.3)$$

where  $a' \equiv da/Nd\tau$  and the integration runs over the period of  $\tau$  on the circle  $S^1$  of  $S^1 \times S^3$ . Here the effective Lagrangian of its local part  $\mathcal{L}(a, a')$  includes the classical Einstein term with the renormalized Planck mass  $m_P = (3\pi/4G)^{1/2}$  and the cosmological constant  $\Lambda = 3H^2$  and contains also the contribution of the conformal anomaly of quantum fields and their vacuum (Casimir) energy,

$$\mathcal{L}(a, a') = m_P^2 \left\{ -aa'^2 - a + H^2 a^3 + B \left( \frac{a'^2}{a} - \frac{a'^4}{6a} + \frac{1}{2a} \right) \right\}, \quad (2.4)$$

$$F(\eta) = \pm \sum_{\omega} \ln(1 \mp e^{-\omega\eta}). \quad (2.5)$$

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<sup>1</sup>In view of the time-parametrization invariance of  $\Gamma[a, N]$  the second equation  $\delta\Gamma/\delta a(\tau) = 0$  is the derivative of the Friedmann equation and should not be imposed independently of the latter [1].

The constant  $B > 0$  is determined by the coefficient of the Gauss-Bonnet term in the overall conformal anomaly of all CFT fields  $\phi$  [9]. A nonlocal part of the action  $F(\eta)$  is the free energy of their quasi-equilibrium excitations with the temperature given by the inverse of the conformal time (2.3). This is a typical boson or fermion sum over field oscillators with energies  $\omega$  on a unit 3-sphere.

Semiclassically the integral (1.3) is dominated by the saddle points — solutions of the effective Friedmann equation of the *Euclidean* gravity theory

$$-\frac{a'^2}{a^2} + \frac{1}{a^2} - B \left( \frac{a'^4}{2a^4} - \frac{a'^2}{a^4} \right) = \frac{\Lambda}{3} + \frac{C}{a^4}, \quad (2.6)$$

$$C = \frac{B}{2} + \frac{1}{m_P^2} \frac{dF(\eta)}{d\eta}, \quad (2.7)$$

modified by the quantum  $B$ -term and the radiation term  $C/a^4$ . The constant  $C$  here characterizes the sum of the renormalized Casimir energy  $B/2$  and the energy of the gas of thermally excited particles

$$\frac{dF(\eta)}{d\eta} = \sum_{\omega} \frac{\omega}{e^{\omega\eta} \mp 1}. \quad (2.8)$$

Thus, the amount of thermal radiation in this self-consistent back reaction problem is not put by hands, but determined by the *bootstrap* equation (2.7) which yields  $C$  as a functional of the FRW geometry in which this radiation evolves.

The inverse temperature  $\eta$  – the instanton period in units of the conformal time – is given by the integral (2.3) over the full period of  $\tau$  or the  $2k$ -multiple of the integral between the two neighboring turning points of the scale factor history  $a(\tau)$ ,  $a'(\tau_{\pm}) = 0$ . This  $k$ -fold nature implies that in the background solution during the full period of its Euclidean time the scale factor oscillates  $k$  times between its maximum and minimum values  $a_{\pm} = a(\tau_{\pm})$ ,  $a_- \leq a(\tau) \leq a_+$ ,

$$a_{\pm}^2 = \frac{1}{2H^2} (1 \pm \sqrt{1 - 4H^2 C}), \quad (2.9)$$

and forms a kind of a garland of  $S^1 \times S^3$  topology with oscillating  $S^3$  sections. As shown in [2], these garland-type instantons exist only in the limited range of the cosmological constant  $\Lambda = 3H^2$ ,  $0 < H_{\min}^2 < H^2 < H_{\max}^2 = 1/2B$  and are weighted in the relevant statistical ensemble by their exponentiated on-shell action according to (1.5), where  $I_0 = I[a, N]$  is taken at the solution of Eqs.(2.6)-(2.7). In particular, a set of instantons with  $k = 1$  – a single oscillation of the scale factor  $a(\tau)$  between its minimal and maximal values (2.9) – occupies a certain continuous range of  $H^2$  inside  $[H_{\min}^2, H_{\max}^2]$  adjacent to its lower boundary [2].

The main effect of these exponential weights is a complete suppression of another set of solutions – the vacuum Hartle-Hawking instantons [10] with no radiation  $dF/d\eta = 0$ , which represent the Euclidean de Sitter spacetime. These purely vacuum contributions are ruled out by their infinite *positive* effective action (cf.  $1/a$ -factor in the kinetic  $B$ -terms of the effective action (2.4) which render its integrand diverging to  $+\infty$  at  $\tau_-$  with  $a \rightarrow 0$  and  $a' \rightarrow 1$ ). Otherwise, these exponential weights are of the same order of magnitude and they are inefficient to select most probable configurations. This makes important the calculation of the one-loop prefactor  $P$  which is the main focus of this paper. The technique for this prefactor was developed for a parametrization invariant system (2.2) with a generic Lagrangian  $\mathcal{L}(a, a')$  restricted only by the order of derivatives of  $a$ .<sup>2</sup> Below we apply this technique to the CFT driven cosmology of the above type for a set of one-fold instantons with  $k = 1$ .

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<sup>2</sup>Note that the local Lagrangian of (2.4) does not contain higher order derivatives of  $a$ . This is the result of the special renormalization [2], which does not introduce into the minisuperspace sector of Einstein theory extra degrees of freedom. This choice was motivated in [2] by ghost-free requirements and certain universality properties which, in particular, relate the value of the Casimir energy in (2.4),  $B/2a$ , to the coefficient  $B$  of the conformal anomaly [11, 12].

Cosmological instantons in the trace anomaly driven cosmology were also considered in [5]. It should be emphasized, however, that setting the problem and the field-theoretical model itself in this work are essentially different from our approach in [2]. To begin with, the solutions of [5] correspond to the Hartle-Hawking no-boundary wavefunction and have the  $S^4$ -topology in contrast to our periodic  $S^1 \times S^3$  instantons associated with the microcanonical statistical sum. Therefore, the amount of radiation in [5] is a freely specifiable constant of integration, while here it is strictly fixed by the “bootstrap” equation (2.7). Secondly, the freedom of UV renormalization is not used in [5] to eradicate higher order derivatives from the Friedmann equation, whereas in [2] this freedom is fixed by the requirement to preserve the classical number of physical degrees of freedom and the stability of the theory against higher-derivative ghosts (cf. footnote 2 above). Finally, a primordial (or renormalized) cosmological constant is assumed to be zero in [5], whereas here  $\Lambda$  belongs to a nontrivial range where it parameterizes the instanton solutions.

As a result the so-called double-bubble instantons of [5] do not exist in our setting. This automatically removes the collapsing universes found in [5] by analytic continuation to the physical Lorentzian time. Instead we have a thermal version of the Hartle-Hawking instantons – the instanton garlands of [2] which give rise to expanding Lorentzian universes with different values of the primordial cosmological constant  $\Lambda$ . As discussed in [4], they contain a rapidly diluting radiation of conformal particles and have a finite inflationary stage terminating with the decay of  $\Lambda$  if the latter has a composite nature of a slowly varying inflaton field. This justifies the use these instantons as initial conditions for the Universe, capable of formation of the observable large scale structure.

Note that in our setting even the vacuum Hartle-Hawking instantons mentioned above originate from the periodic boundary conditions as a limiting case when a torus  $S^3 \times S^1$  gets ripped at the vanishing value of  $a_- = 0$  and topologically becomes a 4-sphere  $S^4$ . Thus, these round sphere instantons exist just like in [5],<sup>3</sup> but as mentioned above they are completely ruled out by their infinite positive Euclidean action  $I_0 = +\infty$  [2] – the diverging contribution of the conformal anomaly. Quite interestingly, this property was not observed in [5] because the action  $I_0$  was not calculated there.

### 3. The one-loop prefactor for a generic time-parametrization invariant model

The one-loop preexponential factor of the statistical sum (1.5) was obtained in [1] for a one-fold instanton background. For a theory with the generic effective action (2.2) it reads as

$$P = \text{const} \times \frac{1}{\sqrt{\left| 1 - \mathbf{I} \frac{d^2 F}{d\eta^2} \right|}} \quad (3.1)$$

where  $-d^2 F/d\eta^2$  is a kind of a specific heat of the thermal bath of matter quanta and  $\mathbf{I}$  is a special functional of the background solution  $(a(\tau), N(\tau))$ . As shown in [1, 13] this quantity also generates the functional determinant

$$\text{Det}_* \mathbf{F} = \text{const} \times \mathbf{I} \quad (3.2)$$

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<sup>3</sup>This follows from the fact that the effective Friedmann equation (2.6) for  $H^2 = 0$  exactly coincides with its analogue (2.18) in [5] under a special choice of the UV counter term  $\sim R^2$  which eradicates higher order derivatives from this equation. But, as mentioned in [5], the inclusion of higher derivatives induced by  $R^2$  does not destroy a round sphere instanton, while adding a nonvanishing cosmological constant only modifies its effective Hubble parameter.

of the operator  $\mathbf{F}$  of small disturbances of the canonically normalized mode of the scale factor perturbation  $\varphi = \sqrt{|\mathcal{D}|} \delta a$  propagating on this background<sup>4</sup>,

$$\mathbf{F} = -\frac{d^2}{d\tau^2} + \frac{g''}{g}, \quad (3.3)$$

$$g = a' a \sqrt{|\mathcal{D}|}, \quad \mathcal{D} = \frac{\partial^2 \mathcal{L}}{\partial a' \partial a'}. \quad (3.4)$$

This operator is expressed via the function  $g = g(\tau)$  which simultaneously serves as its zero mode,  $\mathbf{F}g(\tau) = 0$ , and in its turn expresses in terms of the time derivative of the scale factor  $a'$  and the Hessian of the Lagrangian  $\mathcal{L}(a, a')$ . Thus all the quantities are functionals of this zero mode which itself is easily calculable by Eq.(3.4) for a given instanton background.

The quantity  $\mathbf{I}$  expresses in quadratures as a functional of  $g(\tau)$  as follows. For a one-fold instanton the function  $g(\tau) \propto a'(\tau)$  has one oscillation in the full range of the Euclidean time forming a circle. Therefore, it has two zeroes at the antipodal points of this circle  $\tau_{\pm}$  corresponding to a maximum and minimum values of the scale factor. These points mark the boundaries of the half period of the total time range,  $T = 2(\tau_+ - \tau_-)$ , at which the periodic function  $g(\tau)$  has these two first degree zeros

$$g(\tau_{\pm}) = 0, \quad g'(\tau_{\pm}) \equiv g'_{\pm} \neq 0. \quad (3.5)$$

Then  $\mathbf{I}$  is given by a combination

$$\mathbf{I} = 2\varepsilon_{\mathcal{D}} (\Psi_+ \Psi'_+ - \Psi_- \Psi'_-), \quad \varepsilon_{\mathcal{D}} = \frac{\mathcal{D}}{|\mathcal{D}|} = \pm 1, \quad (3.6)$$

$$\Psi_{\pm} \equiv \Psi(\tau_{\pm}), \quad \Psi'_{\pm} \equiv \Psi'(\tau_{\pm}), \quad (3.7)$$

of boundary values at these points of a special (non-periodic) solution  $\Psi(\tau)$  of the homogeneous equation  $\mathbf{F}\Psi(\tau) = 0$  with the same operator (3.3). This solution is defined by the equation

$$\Psi(\tau) \equiv g(\tau) \int_{\tau_*}^{\tau} \frac{dy}{g^2(y)}, \quad \tau_- \equiv 0 < \tau < \tau_+, \quad \tau_- < \tau_* < \tau_+, \quad (3.8)$$

where  $\tau_*$  is some fixed point belonging to this half period range of  $\tau$ .

The main property of the function  $\Psi(\tau)$  is that it is smoothly defined in the half-period range of  $\tau$  and  $\tau_*$ , where the integral (3.8) is convergent because the roots of  $g(\tau)$  do not occur in the integration range of (3.8). It cannot be smoothly continued beyond this half-period, though its limits are well defined for  $\tau \rightarrow \tau_{\pm} \mp 0$ ,

$$\Psi(\tau_{\pm}) = -\frac{1}{g'(\tau_{\pm})} \equiv -\frac{1}{g'_{\pm}}, \quad (3.9)$$

because the factor  $g(\tau)$  tending to zero compensates for the divergence of the integral at  $\tau \rightarrow \tau_{\pm}$ . Moreover, because of  $g''(\tau_{\pm}) = 0$  the function  $\Psi(\tau)$  is differentiable at  $\tau \rightarrow \tau_{\pm}$ , and all the quantities which enter the algorithm (3.6) are well defined. These properties of  $\Psi(\tau) = \Psi(\tau, \tau_*)$  guarantee that  $\mathbf{I}$  is independent of an arbitrary choice of the point  $\tau_*$ , which can be easily verified by using a simple relation  $d\Psi'_{\pm}/d\tau_* = -g'_{\pm}/g^2(\tau_*)$ .

The structure of (3.1) suggests that the effect of the one-loop prefactor can be strong, because the argument of the square root can tend to zero. Though the factor  $d^2 F/d\eta^2$  which is proportional to the negative of the specific heat of the thermal bath,

$$\frac{d^2 F}{d\eta^2} = -\sum_{\omega} \frac{\omega^2 e^{\omega\eta}}{(e^{\omega\eta} \mp 1)^2} < 0, \quad (3.10)$$

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<sup>4</sup>Star in the notation  $\text{Det}_*$  implies gauging out the zero mode of this operator, originating from the residual conformal Killing symmetry of the action for  $\varphi$  [1, 13].

is always negative, the sign of  $\mathbf{I}$  is indefinite. Indeed, this quantity can be rewritten as the action functional of the mode  $\Psi(\tau)$  in the half period of the Euclidean time range

$$\mathbf{I} = 2\varepsilon_{\mathcal{D}} \int_{\tau_-}^{\tau_+} d\tau \left( \Psi'^2 + \frac{g''}{g} \Psi^2 \right). \quad (3.11)$$

Its sign depends on the sign of the Hessian matrix in (3.6) (ghost or non-ghost nature of scale factor perturbation  $\varphi$  in the gravitational action). Moreover, the integral here is also indefinite, because the potential term is not positive-definite. In fact it is basically negative ( $\varphi$  is a tachyon), because for an oscillating  $g(\tau) \sim a'(\tau)$  this function is quasi-harmonic with  $g''/g < 0$  in the average over the oscillation period. In particular, for a constant  $g''/g = -w^2$ , the function  $g(\tau)$  is harmonically oscillating, the function  $\Psi(\tau)$  is exactly calculable and  $\mathbf{I}$  identically vanishes – the case of the second zero mode of  $\mathbf{F}$  complimentary to  $g(\tau)$  [13],

$$g(\tau) = d \sin(w\tau), \quad \Psi(\tau) = \frac{\sin(w(\tau - \tau_*))}{d \sin(w\tau_*)}, \quad \Psi_+ \Psi'_+ - \Psi_- \Psi'_- = 0. \quad (3.12)$$

Therefore, a nontrivial expression for the prefactor is based on the anharmonicity of  $g(\tau)$  and should be analyzed numerically or by some kind of perturbation theory.

## 4. Estimates for the prefactor in the CFT driven cosmology

Now we apply the above technique to the model of the CFT driven cosmology described in Introduction. As was shown in [2], all saddle-point instantons of this model exist only inside the curvilinear wedge in the two-dimensional plane of the variables  $(H^2, C)$  – primordial cosmological constant  $\Lambda = 3H^2$  and the amount of radiation constant  $C$ , depicted in Fig.1. This wedge is bounded by the lower straight-line and upper hyperbolic boundaries,

$$B - B^2 H^2 < C < 1/4H^2. \quad (4.1)$$

The one-parameter families of  $k$ -fold instantons form in this plane the curves interpolating between these two boundaries, and their sequence accumulates at the critical point corresponding to the quantum gravity scale  $H_{\max}^2 = 1/2B$ . Each instanton solution is represented by a point on one of these curves with the corresponding  $k$  – the number of oscillations of the scale factor  $a$  between its minimal and maximal values. The range of  $\Lambda$  in which these  $k$ -fold instantons exist is given by a projection of their family on the axes of the variable  $H^2$ . Here we consider only the one-fold family.

Analytic estimates of the prefactor are hard to make even in the limit of a large number of quantum fields  $N \gg 1$  and a large value of  $B$ , because as we will now see this one-loop factor is  $O(1)$  and is not sensitive to this limit. A natural quantity that could have played the role of a smallness parameter is the combination

$$\varepsilon = 1 - 2BH^2, \quad (4.2)$$

which belongs to the bounded range  $0 \leq \varepsilon \leq 1$  for all these instantons. As was shown in [2] for multi-fold instantons with  $k \gg 1$  this quantity is indeed small  $\varepsilon \simeq \ln k^2 / 2\pi^2 k^2 \rightarrow 0$  and corresponds to solutions tending to the corner of the curvilinear triangle  $C = B/2$  – the point of a new quantum gravity scale  $H^2 = 1/2B$ . Unfortunately, for the case of  $k = 1$  instanton which only we consider here  $\varepsilon = O(1)$  and the corresponding prefactor should be calculated numerically.

To make this calculation manageable let us first parameterize the instanton solutions in a more convenient way in terms of the quantity (4.2). From the modified Friedmann equation (2.6) we know that the scale factor is a function of time oscillating between the two turning points  $a_{\pm}$ . In terms of a dimensionless variable

$$z = \frac{a^2}{B} \quad (4.3)$$

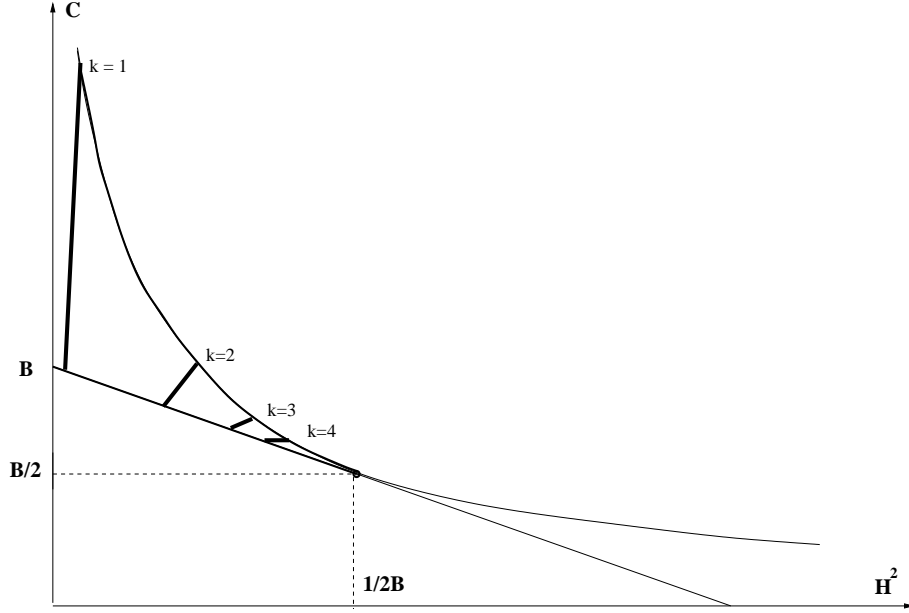


Figure 1: Instanton domain in the  $(H^2, C)$ -plane. Instanton families are shown for  $k = 1, 2, 3, 4$ . Their sequence accumulates at the critical point  $(1/2B, B/2)$ .

this behavior can be expressed as a solution of (2.6) for the time derivative of  $a(\tau)$

$$a'^2 = (1 - \varepsilon) \frac{(z_+ - z)(z - z_-)}{\sqrt{\varepsilon(z^2 - z_c^2) + z - 1}}, \quad (4.4)$$

where the turning points  $z_{\pm}$  and  $z_c = a_c^2/B$  read in terms of  $B, C$  and  $\varepsilon$  as

$$z_{\pm} = \frac{1}{1 - \varepsilon} \left( 1 \pm \sqrt{1 - \frac{2C}{B}(1 - \varepsilon)} \right), \quad (4.5)$$

$$z_c^2 = \frac{a_c^4}{B^2} = \frac{1}{\varepsilon} \left( \frac{2C}{B} - 1 \right). \quad (4.6)$$

In terms of  $\varepsilon$  the curvilinear wedge (4.1) where the instantons are located looks as

$$\varepsilon < \frac{2C}{B} - 1 < \varepsilon + \frac{\varepsilon^2}{1 - \varepsilon}. \quad (4.7)$$

Inside this domain the one-parameter family of solutions interpolating between its boundaries can be labeled by the parameter  $d$  ranging from  $d = 0$  at the upper boundary to  $d = 1$  at the lower boundary

$$\frac{2C}{B} - 1 = \varepsilon + (1 - d^2) \frac{\varepsilon^2}{1 - \varepsilon}, \quad 0 \leq d \leq 1. \quad (4.8)$$

Together with the definition (4.2) this relation is just a relabeling of the solutions from the dimensional variables  $(C, H^2)$  to the dimensionless  $(\varepsilon, d)$  having a simple finite range inside a unit quadrangle.

In terms of  $d$  the turning points of the solution  $z_{\pm}$  and the parameter  $z_c$  introduced above take the form

$$z_{\pm} = \frac{1 \pm \varepsilon d}{1 - \varepsilon}, \quad z_c^2 = \frac{1 - \varepsilon d^2}{1 - \varepsilon} \quad (4.9)$$

With  $z$  parameterized between the two turning points by the new variable  $x$

$$z = z_- + \frac{2\varepsilon d}{1-\varepsilon} x, \quad 0 \leq x \leq 1 \quad (4.10)$$

the equation for  $a'$  can be written down as

$$a'^2 = 4\varepsilon d^2 \frac{x(1-x)}{[(1-d)^2 + 4d(1-\varepsilon d)x + 4\varepsilon d^2 x^2]^{1/2} + 1 - d + 2dx}, \quad (4.11)$$

so that the conformal time period of the instanton equals

$$\begin{aligned} \eta = 2 \int_{a_-}^{a_+} \frac{da}{a'a} &= \sqrt{\varepsilon} \int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}} \\ &\times \frac{([ (1-d)^2 + 4d(1-\varepsilon d)x + 4\varepsilon d^2 x^2 ]^{1/2} + 1 - d + 2dx)^{1/2}}{(1 - \varepsilon d + 2\varepsilon^2 dx)}. \end{aligned} \quad (4.12)$$

With a very good accuracy for all  $\varepsilon$  and  $d$  the integral here equals  $\pi\sqrt{2}$ , so that  $\eta \simeq \pi\sqrt{2\varepsilon}$ . On the other hand, the bootstrap equation (2.7) for  $\varepsilon$  in view of (4.8) reads

$$\frac{2C}{B} - 1 \equiv \varepsilon + (1-d^2) \frac{\varepsilon^2}{1-\varepsilon} = \frac{2}{m_P^2 B} \frac{dF}{d\eta}. \quad (4.13)$$

In view of the fact that both  $B \sim \mathbb{N}$  and  $dF/d\eta \sim \mathbb{N}$  for a large number of species  $\mathbb{N}$ , the ratio on the right hand side here is  $O(\mathbb{N}^0) = O(1)$ , and the parameter  $\varepsilon$  also remains  $O(1)$  and is not sensitive to the limit of large  $\mathbb{N}$ . Then one can easily check that in the limits of either small,  $\eta \ll 1$ ,  $dF/d\eta \sim 1/\eta^4$ , or relatively large,  $\eta \sim \pi$ ,  $dF/d\eta \sim \exp(-\#\eta)$ , values of  $\eta$  this equation yields the value of  $\varepsilon$  close to the middle of its admissible range  $0 \leq \varepsilon \leq 1$ ,  $\varepsilon \sim 1/2$ . This confirms that for a one-fold garland instanton  $\varepsilon$  never gets either very small or close to one, and numerical simulations become inevitable for the calculation of the prefactor (3.1) with its basic quantity (3.6). Fortunately, this leads to a simple qualitative picture.

The function (3.4) for the Lagrangian of (2.4) equals in terms of the variable  $z$ , (4.3), and the running integration parameter  $x$  of Eq.(4.10)

$$g^2 = 12\pi^2 M_P^2 B a a'^2 \sqrt{\varepsilon (z^2 - z_c^2)}, \quad (4.14)$$

$$\sqrt{z^2 - z_c^2} = \frac{\sqrt{\varepsilon}}{1-\varepsilon} [(1-d)^2 + 4d(1-\varepsilon d)x + 4\varepsilon d^2 x^2]^{1/2}. \quad (4.15)$$

Therefore the function (3.8) is given by the following integral

$$\Psi(\tau) = g(\tau) \int_{a_*}^{a(\tau)} \frac{da}{a' g^2} = \frac{g(\tau)}{16 m_P^2 B} \frac{1}{d^2} \int_{x_*}^{x(\tau)} \frac{dy}{y^{3/2} (1-y)^{3/2}} G(y), \quad (4.16)$$

$$\begin{aligned} G(y) &= \frac{1}{\varepsilon^{3/2}} \left( \frac{1-\varepsilon}{1-\varepsilon d + 2\varepsilon^2 dy} \right)^{3/2} \\ &\times \frac{([ (1-d)^2 + 4d(1-\varepsilon d)y + 4\varepsilon d^2 y^2 ]^{1/2} + 1 - d + 2dy)^{3/2}}{[(1-d)^2 + 4d(1-\varepsilon d)y + 4\varepsilon d^2 y^2]^{1/2}}, \end{aligned} \quad (4.17)$$

where the integration limits equal

$$x(\tau) = \frac{1-\varepsilon}{2\varepsilon d} \frac{a^2(\tau) - a_-^2}{\varepsilon B}, \quad x_* = \frac{1-\varepsilon}{2\varepsilon d} \frac{a_*^2 - a_-^2}{\varepsilon B}. \quad (4.18)$$



In the limit  $\tau \rightarrow \tau_{\pm}$  (equivalent to  $x(\tau) \rightarrow 1$  and  $x(\tau) \rightarrow 0$ ) the integral in (4.16) is divergent, this divergence being compensated by the factor  $g(\tau)$  vanishing at these points. For the calculation of (3.6) we have to take these limits, and their calculation is based on a preliminary integration by parts in  $y$  which disentangles the divergent part of the integral,  $\sim 1/\sqrt{1-x(\tau)}$  at  $x(\tau) \rightarrow 1$  or  $\sim 1/\sqrt{x(\tau)}$  at  $x(\tau) \rightarrow 0$ . Then multiplying the result by  $g(\tau)$  and differentiating we get in view of (3.9) (we take into account that for the CFT driven cosmology  $\varepsilon_{\mathcal{D}} = -1$ )

$$\mathbf{I} = -2(\Psi_+ \Psi'_+ - \Psi_- \Psi'_-) = -\frac{G}{8d^2 m_P^2 B}, \quad (4.19)$$

$$G = \int_{x_*}^1 dy \frac{G(y) - G(1) y^{3/2}}{y^{3/2}(1-y)^{3/2}} + \int_0^{x_*} dy \frac{G(y) - G(0)(1-y)^{3/2}}{y^{3/2}(1-y)^{3/2}} - \frac{2G(1)}{(1-x_*)^{1/2}} - \frac{2G(0)}{x_*^{1/2}}. \quad (4.20)$$

This expression is, of course, independent of the choice of the intermediate point  $x_*$ , as is easily seen by differentiation with respect to  $x_*$ .

For  $d \rightarrow 0$  the function  $G = O(d^2)$ , so that  $\mathbf{I}$  is finite in this limit even despite  $d^2$  in the denominator of (4.19). Note that the limit  $d \rightarrow 0$  corresponds to the upper hyperbolic boundary of the instanton domain (4.1) on which the cosmological model is static,  $a(\tau) = a_{\pm} = 1/H\sqrt{2}$ , and  $g''/g = -2(1 - \varepsilon)/B\varepsilon = -w^2$  is constant. So this is the case of the harmonic function  $g(\tau)$  in Eq.(3.12) for which formally  $\mathbf{I} = 0$  – the case of the second zero mode of the operator  $\mathbf{F}$ , discussed in [13]. However, in this limit not only the deviation from harmonicity  $O(d^2)$  disappears, but also the amplitude of the function  $g(\tau) \sim d$  tends to zero, cf. Eq.(4.14). Therefore, in view of (3.12) the function  $\mathbf{I}(d) = O(d^2)/d^2$  acquires a  $1/d^2$  factor, and does not vanish for  $d \rightarrow 0$ . Thus, quantum corrections to the preexponential factor are nontrivial even in this limit. As we will see below, they are smaller than those of  $d \rightarrow 1$  but still nonzero.

Numerical calculation of (4.20) can be approximately done in the full range of  $d$ . First we find from the bootstrap equation (4.13) the value of  $\varepsilon$  as a function of  $d$  on the relevant one-parameter family of instantons  $0 \leq d \leq 1$ ,  $\varepsilon = \varepsilon(d)$ , and then evaluate with the aid of (4.19), (4.20) and (3.10) the preexponential factor on this family

$$P(d) = \left| 1 - \mathbf{I} \frac{d^2 F}{d\eta^2} \right|_{\varepsilon=\varepsilon(d)}^{-1/2}. \quad (4.21)$$

We will explicitly demonstrate this procedure on the example of a single gauge vector field. For this field the trace anomaly coefficient  $B$  and the sum (2.8) over its oscillator frequencies  $\omega = \sqrt{2n^2 - 4}$ ,  $n \geq 2$ , read as

$$m_P^2 B = \frac{31}{120}, \quad \frac{dF}{d\eta} = 2 \sum_{n=2}^{\infty} (n^2 - 1) \frac{\sqrt{2n^2 - 4}}{e^{\eta \sqrt{2n^2 - 4}} - 1}. \quad (4.22)$$

With these expressions and the conformal time  $\eta$  given by the integral (4.12) the numerical solution of the bootstrap equation (4.13) can be sufficiently accurately fitted by the following function

$$\varepsilon_{\text{vector}}(d) = 0.345 - 0.096 d + 0.042 d^2, \quad (4.23)$$

which parameterizes the one-fold family of instantons. Then we compute  $\mathbf{I}$  via (4.19)-(4.20), which turns out to be negative on this family, and also evaluate the specific heat for the free energy of vector particles on a 3-sphere

$$\frac{d^2 F}{d\eta^2} = -2 \sum_{n=2}^{\infty} (n^2 - 1) \frac{(2n^2 - 4) e^{\eta \sqrt{2n^2 - 4}}}{(e^{\eta \sqrt{2n^2 - 4}} - 1)^2}. \quad (4.24)$$

The use of (4.21) for this family of solutions finally yields the plot of  $P(d)$  depicted on Fig.2. A characteristic feature of this vector field case is that with  $d^2 F/d\eta^2 < 0$  and negative  $\mathbf{I}$  the denominator of  $P(d)$  is smaller than one, but never degenerates to zero. Therefore, the one-loop prefactor stays slightly higher than one, but remains finite and within the perturbation theory range.

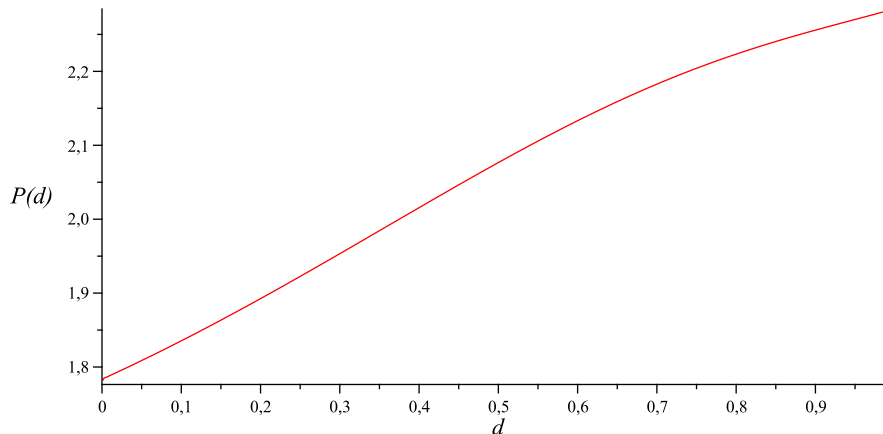


Figure 2: Plot of  $P(d)$  for for a gauge vector field.

Similar calculations can be done for pure massless spinor and conformal scalar fields. The prefactor for these fields remains nearly constant throughout the whole family of one-fold instantons. For a spinor field the behavior of  $P(d) \simeq 1.1 > 1$  is qualitatively the same as in the vector field case, whereas for a scalar field the calculations show that  $\mathbf{I} > 0$  and the prefactor turns out to be smaller than one,  $P(d) \simeq 0.9 < 1$ . Possible mixtures of various spins do not qualitatively change this picture.

## 5. Conclusions

The one-loop prefactor of the statistical sum in the CFT driven cosmology was obtained for a limited set of saddle-point solutions having one oscillation of the cosmological scale factor. This limitation is currently explained by the fact that the major ingredient of the calculational algorithm – the restricted functional determinant of the operator  $\mathbf{F}$  is known in a closed form (3.2) only for this simplest set of instantons [13]. The resulting preexponential factor does not present us with new physics, because for all scalar, spinor and vector conformal particles or their combinations the prefactor remains a smooth function  $P = O(1)$ . It justifies the semiclassical expansion, but does not feature phase transitions or strong coupling range beyond perturbation theory restrictions even in the limit of numerous conformal species  $\mathbb{N} \gg 1$ . At the same time path integral applications in such conformal cosmology suggest the existence of instantons with an arbitrary number of oscillations of the cosmological scale factor. In particular, for the number of these oscillations tending to infinity this CFT driven cosmology approaches a new quantum gravity scale  $\Lambda_{\max} = 3H_{\max}^2 = 3/2B$  – the maximum possible value of the cosmological constant [2, 3] – where the physics and, in particular, the effects of the quantum prefactor become very interesting and important. Thus the extension of the above results to the full set of cosmological instantons looks very promising, as this extension might be relevant to the resolution of the cosmological constant and landscape problems [3]. We hope to attain this extension in the foreseeable future.

Main difficulty with all known applications of quantum gravity and cosmology is that the lack of UV complete theory forces us to work within the effective field theory trustable only below its cutoff. In this respect large  $\mathbb{N}$  CFT driven cosmology of [2] seems to be a safe theory because its instanton

curvature scale is bounded by  $H_{\text{max}}^2 = 1/2B \sim m_P^2/\mathbb{N} \ll m_P^2$  which is much lower than the Planck scale. Therefore, no non-perturbative extension (like the hypothetical asymptotic safety of quantum gravity [14]) is likely to be needed, and a usual semiclassical expansion is sufficient to handle the problem of cosmological initial conditions. However, as persuasively advocated in [15], the cutoff in theories with a large number of quantum species,  $\sim m_P^2/\mathbb{N}$ , also decreases with the growing  $\mathbb{N}$  and the instantons considered above turn out to be very close to this cutoff scale. This makes important the computation of loop corrections initiated above. Finiteness of the one-loop order and its regularity  $P = O(1)$  in the full one-fold instanton range of  $\Lambda$  gives a hope that this property will survive beyond this range and beyond one-loop approximation, and this is a subject of our further studies [16].

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## References

- [1] A. O. Barvinsky, *The path integral for the statistical sum of the microcanonical ensemble in cosmology*, arXiv:1012.1568 [hep-th].
- [2] A. O. Barvinsky and A. Yu. Kamenshchik, JCAP 09 (2006) 014, arXiv:hep-th/0605132; Phys. Rev. D **74** (2006) 121502, arXiv:hep-th/0611206.
- [3] A. O. Barvinsky, Phys. Rev. Lett. **99** (2007) 071301, arXiv:hep-th/0704.0083.
- [4] A. O. Barvinsky, C. Deffayet and A. Yu. Kamenshchik, JCAP 05 (2008) 020, arXiv:0801.2063.
- [5] S. W. Hawking, T. Hertog and H. S. Reall, Phys. Rev. D **63** (2001) 083504, hep-th/0010232.
- [6] M. V. Fischetti, J. B. Hartle and B. L. Hu, 1979 Phys. Rev. D **20** 1757.
- [7] A. A. Starobinsky, 1980 Phys. Lett. B **91** 99.
- [8] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. **B 134**, 187 (1984); A. O. Barvinsky, A. G. Mirzabekian and V. V. Zhytnikov, gr-qc/9510037; P. O. Mazur and E. Mottola, Phys. Rev. D **64** (2001) 104022.
- [9] M. J. Duff, Class. Quant. Grav. **11** (1994) 1387; M. J. Duff and J. T. Liu, Phys. Rev. Lett. **85** (2000) 2052.
- [10] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28** (1983) 2960; S. W. Hawking, Nucl. Phys. B **239** (1984) 257.
- [11] L. S. Brown and J. P. Cassidy, Phys. Rev. D **16** (1977) 1712; I. Antoniadis, P. O. Mazur and E. Mottola, Phys. Rev. D **55** (1997) 4770, arXiv:9509169[hep-th].
- [12] A. O. Barvinsky, C. Deffayet and A. Yu. Kamenshchik, JCAP 05 (2010) 034, arXiv:0912.4604
- [13] A. O. Barvinsky and A. Yu. Kamenshchik, *On the functional determinant of a special operator with a zero mode in cosmology*, arXiv:1012.1571 [hep-th].

- [14] S. Weinberg, *Critical Phenomena for Field Theorists*, in *Understanding the Fundamental Constituents of Matter*, ed. A. Zichichi (Plenum Press, New York, 1977), Phys. Rev. **D 81** (2010) 083535, arXiv:0911.3165 [hep-th]; C. Wetterich, Phys. Lett. **B 301** (1993) 90.
- [15] G.Dvali, Fortsch. Phys. **58** (2010) 528, arXiv:0706.2050 [hep-th]; G. Dvali and M. Redi, Phys. Rev. **D77** (2008) 045027, arXiv:0710.4344 [hep-th].
- [16] A. O. Barvinsky, Yu. V. Gusev and D. V. Nesterov, work in progress.